Differential Geometry of Surfaces with Mathcad: A Virtual Learning Approach

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Abstract

In this paper we propose an alternative to traditional teaching techniques of Differential Geometry. The new concept is to create a virtual learning environment by using modern software with good capabilities for plotting curves and surfaces. For this purpose we used Mathcad because this software has a user friendly interface in which it is easy to combine math equations, plots and texts.

Keywords: Differential Geometry, surfaces, tangent plane, Mathcad.

1. Introduction

Teaching Differential Geometry of surfaces for students in engineering is a difficult task for every teacher, because this topic requires not only that the students have solid knowledge of geometry, calculus and linear algebra but they must also have a good 3D imagination. The Differential Geometry requires the use of visual tools for better understanding, because it is three–dimensional geometry with high complexity degree. Traditionally, for the study of a surface the teacher draws on the blackboard the surface, the tangent planes and normal lines at some point of the surface, some curves on surface and the angles between them etc.

In this paper we propose an alternative to traditional teaching techniques of Differential Geometry. The new concept is to create a virtual learning environment by using modern software with good capabilities for plotting curves and surfaces. For this purpose we used Mathcad because it has a user friendly interface in which it is easy to combine math equations, plots and texts. The models initially created by teacher for his lectures can be later used by students for the visualization of new surfaces or for computation of some numerical characteristic associated to the surfaces. All these facts are possible because the environment is an interactive Mathcad e-book in which the students can make their own changes and can see immediately the answer to these modifications.

Section 2 contains some theoretical background about the surfaces. This section is necessary especially for recalling the formulas used in the rest of the paper. Section 3 contains an example. To show the possibility offered by the techniques base on Mathcad for teaching Differential Geometry of surfaces we choose to study a simple surface: the elliptic paraboloid. In Section 4 there are some short conclusions.

2. Surfaces in Space: A Theoretical Background

A set $D \subset \mathbf{R}^2$ is called a *domain* if it is open and connected. The domain D is called an *elementary domain* if it is homeomorphic to an open disk. (A homeomorphism from a geometric figure to another is a one-to-one map that is continuous and has continuous inverse.)

A set S in space is called an *elementary surface* if it is the image of a planar elementary domain D under a homeomorphism $\vec{\mathbf{r}} : D \to \mathbf{R}^3$. If we fix in $\mathbf{R}^3$ the canonical orthogonal basis $\{\vec{\mathbf{i}}, \vec{\mathbf{j}}, \vec{\mathbf{k}}\}$ then the surface S has the parametric vector equation

$$[1] \qquad \vec{\mathbf{r}}(u,v) = x(u,v)\,\vec{\mathbf{i}} + y(u,v)\,\vec{\mathbf{j}} + z(u,v)\,\vec{\mathbf{k}}, \quad (u,v) \in D.$$

The pair of the real numbers (u,v) is called the *curvilinear coordinates* of the point $P(x,y,z)$ on the surface. In what follows we assume that the functions $x(u,v), y(u,v)$ and $z(u,v)$ are of class C^1 on D. Such a surface is called *smooth*.

If in equation [1] we take $v = v_0$ as a constant and let u varying, then we obtain a space curve on the surface S,

$$[2] \qquad \Gamma_u(v = v_0): \quad \vec{\boldsymbol{\rho}}_1(u) = \vec{\mathbf{r}}(u,v_0) = x(u,v_0)\,\vec{\mathbf{i}} + y(u,v_0)\,\vec{\mathbf{j}} + z(u,v_0)\,\vec{\mathbf{k}},$$

called the coordinate *u – curve*. The derivate vector

$$[3] \qquad \frac{d\vec{\boldsymbol{\rho}}_1}{du}(u_0) = \frac{\partial \vec{\mathbf{r}}}{\partial u}(u_0,v_0) = \vec{\mathbf{r}}_u(u_0,v_0)$$

is the tangent vector at the curve Γ_u at the point $P_0(x_0 y_0, z_0)$, where $x_0 = x(u_0,v_0)$, $y_0 = y(u_0,v_0)$, $y_0 = y(u_0,v_0)$. Similarly, for $u = u_0$ and v varying, we obtain the coordinate *v – curve* on S,

$$[4] \qquad \Gamma_v(u = u_0): \quad \vec{\boldsymbol{\rho}}_2(v) = \vec{\mathbf{r}}(u_0,v) = x(u_0,v)\,\vec{\mathbf{i}} + y(u_0,v)\,\vec{\mathbf{j}} + z(u_0,v)\,\vec{\mathbf{k}},$$

and the tangent vector at the curve Γ_v at the point $P_0(x_0 y_0, z_0)$

$$[5] \qquad \frac{d\vec{\boldsymbol{\rho}}_2}{dv}(v_0) = \frac{\partial \vec{\mathbf{r}}}{\partial v}(u_0,v_0) = \vec{\mathbf{r}}_v(u_0,v_0).$$

We assume that the tangent vectors $\vec{\mathbf{r}}_u(u,v)$ and $\vec{\mathbf{r}}_v(u,v)$ are *linearly independent* at every point (u,v) belonging to D. This is equivalent with the fact that the cross product $\vec{\mathbf{N}} = \vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v$ is nonzero at every point $(u,v) \in D$.

The plane through point $P_0(x_0 y_0, z_0)$ parallel to vectors $\vec{\mathbf{r}}_u(u_0,v_0)$ and $\vec{\mathbf{r}}_v(u_0,v_0)$ is called the *tangent plane* to the surface S at P_0. This plane is denoted by $T_{P_0}(S)$ and has the vector equation

$$[6] \qquad \vec{\mathbf{T}}(a,b) = \vec{\mathbf{r}}(u_0,v_0) + a\,\vec{\mathbf{r}}_u(u_0,v_0) + b\,\vec{\mathbf{r}}_v(u_0,v_0), \quad a,b \in \mathbf{R}.$$

The vector

[7] $$\vec{\mathbf{N}}(u,v)=\vec{\mathbf{r}}_u(u,v)\times\vec{\mathbf{r}}_v(u,v)\,,\quad (u,v)\in D\,,$$

is called the *normal vector* to the surface S at the point $P(u,v)$. The straight line through the point $P_0(x_0 y_0, z_0)$ of the surface S orthogonal to the tangent plane $T_{P_0}(S)$ is called the *normal line* to the surface S at point P_0. The vector equation of the normal line is

[8] $$\vec{\mathbf{L}}(t)=\vec{\mathbf{r}}(u_0,v_0)+t\,\vec{\mathbf{N}}(u_0,v_0)\,,\quad t\in\mathbf{R}\,.$$

An arbitrary *curve* Γ *on the surface* S is locally defined by equations for the curvilinear coordinates $u=u(t)$, $v=v(t)$, with t in a real interval I. The vector equation of the curve Γ is

[9] $$\vec{\boldsymbol{\rho}}(t)=\vec{\mathbf{r}}(u(t),v(t))\,,\quad t\in I\,.$$

The length of the curvilinear segment situated on the curve $\vec{\boldsymbol{\rho}}(t)$ between the points $M_1(t=t_1)$ and $M_2(t=t_2)$ is computed with the formula

[10] $$L(M_1M_2)=\int_{t_1}^{t_2}\left|\vec{\boldsymbol{\rho}}'(t)\right|dt\,.$$

For unexplained notions about surfaces see (Rovenski, 2000) and (Lipschutz, 1969).

3. A Case Study: The Elliptic paraboloid

The elliptic paraboloid of semi-axis a and b has the equation

$$z=\frac{1}{2}\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right).$$

A simple way to obtain parametric equations for this surface is to put $x=u$, $y=v$ and

$$z=\frac{1}{2}\left(\frac{u^2}{a^2}+\frac{v^2}{b^2}\right),$$

or, $x=\sqrt{2}\,a\,u$, $y=\sqrt{2}\,b\,v$ and $z=u^2+v^2$. Thus the vector equation of the elliptic paraboloid is

[11] $$\vec{\mathbf{r}}(u,v)=\sqrt{2}\,a\,u\,\vec{\mathbf{i}}+\sqrt{2}\,b\,v\,\vec{\mathbf{j}}+(u^2+v^2)\vec{\mathbf{k}}\,,\quad u,v\in\mathbf{R}\,.$$

The plot of this surface using this parameterization is shown in Figure 1. (For all the plot of the elliptic paraboloid we will use the values $a=1$ and $b=1$ for semi-axis.)

A better parameterization for plotting this surface is given by

[12] $$\vec{\mathbf{r}}(u,v)=\sqrt{2}\,a\,\cos(v)\,\vec{\mathbf{i}}+\sqrt{2}\,b\sin(v)\,\vec{\mathbf{j}}+u^2\,\vec{\mathbf{k}}\,,\quad u\in\mathbf{R},\,v\in[0,2\pi)\,.$$

See Figure 2 for an elliptic paraboloid plotted using this equation.

The equation of the elliptic paraboloid must be defined in Mathcad in the form:

$$r(u,v) := \begin{pmatrix} \sqrt{2}\cdot a\cdot u\cdot \cos(v) \\ \sqrt{2}\cdot b\cdot u\cdot \sin(v) \\ u^2 \end{pmatrix}$$

We now define a point P_0 on the surface. To plot this point we use the Mathcad function “CreateSpace” defined for a constant vector function $P(t)$.

$$u_0 := 3 \qquad v_0 := \frac{\pi}{4} \qquad P_0 := r\left(u_0, v_0\right) \qquad {P_0}^T = (3 \ \ 3 \ \ 9) \qquad P(t) := P_0$$

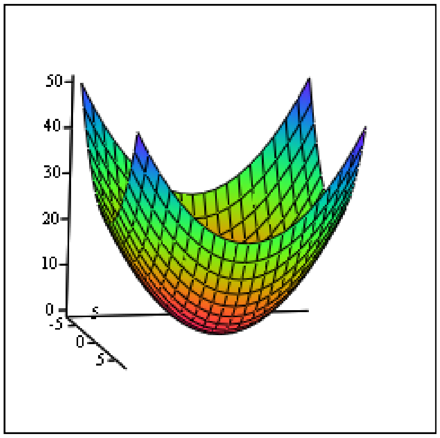
$$P0 := \mathrm{CreateSpace}(P)$$

The coordinate curves which pass through the point P_0 are defined by formulas [2] and [4]. They are plotted by using “CreateSpace” function.

$$\rho 1(u) := r\left(u, v_0\right) \qquad \rho 2(v) := r\left(u_0, v\right)$$

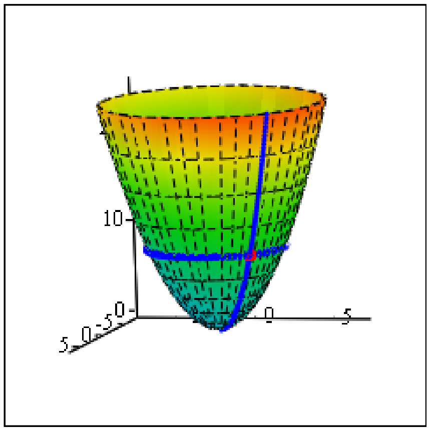
$$\Gamma v := \mathrm{CreateSpace}(\rho 2, -4, 4, 100) \qquad \Gamma u := \mathrm{CreateSpace}(\rho 1, 0, 5, 100)$$

Attention! First plot the point and the coordinate curves and then the surface. For the first three plots use the option “3D Scatter Plot” and for the last plot use the option “Surface Plot” from Graph menu. Figure 2 shows the two coordinates curves on the surface.



r

Figure 1



P0 , Γu , Γv , r

Figure 2

To plot the tangent plane and normal line we first define the derivatives of the vector function $\vec{\mathbf{r}}(u,v)$, that is, the vectors $\vec{\mathbf{r}}_u(u,v)$ and $\vec{\mathbf{r}}_v(u,v)$, and compute their values by using symbolic computation.

$$r_u(u,v) := \begin{pmatrix} \frac{d}{du} r(u,v)_1 \\ \frac{d}{du} r(u,v)_2 \\ \frac{d}{du} r(u,v)_3 \end{pmatrix} \qquad r_v(u,v) := \begin{pmatrix} \frac{d}{dv} r(u,v)_1 \\ \frac{d}{dv} r(u,v)_2 \\ \frac{d}{dv} r(u,v)_3 \end{pmatrix}$$

Now we can define the normal vector at the surface

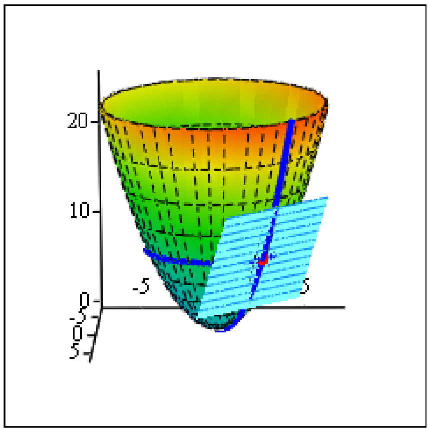
$$N(u,v) := r_u(u,v) \times r_v(u,v)$$

The tangent plane and the normal line to the surface at the given point P_0 have the equations given by the formulas [6] and [8], respectively.

$$T(a,b) := r(u_0,v_0) + a \cdot r_u(u_0,v_0) + b \cdot r_v(u_0,v_0)$$

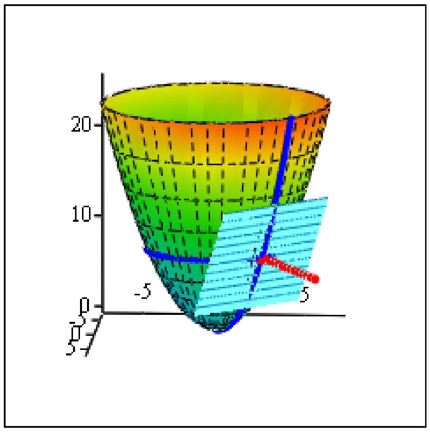
$$L(t) := r(u_0,v_0) + t \cdot N(u_0,v_0)$$

Figures 3 and 4 show the tangent plane and the normal line to the surface at the given point.



P0 , Γu , Γv , r , T

Figure 3



P0 , Γu , Γv , L , r , T

Figure 4

Let us now consider the following two curves which pass through the point (u_0, v_0) in the planar domain of definition of the surface. Figure 5 shows the graphs of these curves.

$$u1(t) := u_0 - t \qquad v1(t) := v_0 + t^3$$

$$u2(t) := u_0 + t^3 \qquad v2(t) := v_0 - t$$

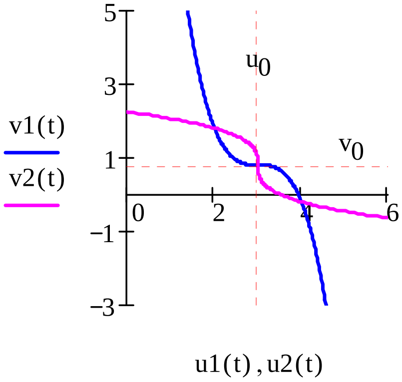


Figure 5

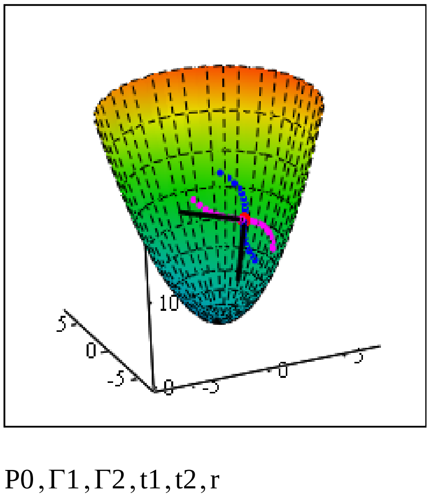


Figure 6

Then we define the two corresponding spatial curves situated on the elliptic paraboloid.

$$\rho1(t) := r(u1(t), v1(t)) \qquad \rho2(t) := r(u2(t), v2(t))$$

For plotting these curves we use the Mathcad function "CreateSpace".

$$\Gamma1 := \mathrm{CreateSpace}(\rho1, -0.7, 0.7) \qquad \Gamma2 := \mathrm{CreateSpace}(\rho2, -0.7, 0.7)$$

In order to plot and to compute the angle between these two curves we define the derivative vectors of the curves,

$$\rho1'(t) := \begin{pmatrix} \frac{d}{dt}\rho1(t)_1 \\ \frac{d}{dt}\rho1(t)_2 \\ \frac{d}{dt}\rho1(t)_3 \end{pmatrix} \qquad \rho2'(t) := \begin{pmatrix} \frac{d}{dt}\rho2(t)_1 \\ \frac{d}{dt}\rho2(t)_2 \\ \frac{d}{dt}\rho2(t)_3 \end{pmatrix}$$

and then the tangent lines to curves at the point (u_0, v_0),

$$\mathrm{t1}(t) := \rho 1(0) + t\cdot\rho 1'(0) \qquad \mathrm{t2}(s) := \rho 2(0) + s\cdot\rho 2'(0)$$

The curves on the surface and the angle between them are shown in Figure 6.

This picture suggests that the angle between these two curves at the given point is equal with $\pi/2$. A simple computation confirms this observation.

$$\theta := \operatorname{acos}\left(\frac{\rho 1'(0)\cdot\rho 2'(0)}{\left|\rho 1'(0)\right|\cdot\left|\rho 2(0)\right|}\right) \qquad \theta \rightarrow \frac{1}{2}\cdot\pi$$

We can also easily calculate the lengths of these two curves.

$$\int_{-0.7}^{0.7} \left|\rho 2'(t)\right|\,dt = 7.877 \qquad \int_{-0.7}^{0.7} \left|\rho 1'(t)\right|\,dt = 9.422$$

Now we consider the following three curves on the elliptic paraboloid

$$\mathrm{u1}(t) := t \quad \mathrm{v1}(t) := 0 \quad \rho 1(t) := r(\mathrm{u1}(t), \mathrm{v1}(t)) \quad C1 := \mathrm{CreateSpace}\left(\rho 1, 0, 3\cdot\sqrt{2}, 100\right)$$

$$\mathrm{u2}(t) := t \quad \mathrm{v2}(t) := \frac{\pi}{4} \quad \rho 2(t) := r(\mathrm{u2}(t), \mathrm{v2}(t)) \quad C2 := \mathrm{CreateSpace}\left(\rho 2, 0, 3\cdot\sqrt{2}, 100\right)$$

$$\mathrm{u3}(t) := 3\cdot\sqrt{2} \quad \mathrm{v3}(t) := t \quad \rho 3(t) := r(\mathrm{u3}(t), \mathrm{v3}(t)) \quad C3 := \mathrm{CreateSpace}\left(\rho 3, 0, \frac{\pi}{4}, 50\right)$$

These curves determine a curvilinear triangle on the paraboloid as we can view in Figure 7.

To compute the perimeter of this curvilinear triangle we define the derivative vectors of every curve.

$$\rho 1'(t) := \begin{pmatrix} \frac{d}{dt}\rho 1(t)_1 \\ \frac{d}{dt}\rho 1(t)_2 \\ \frac{d}{dt}\rho 1(t)_3 \end{pmatrix} \qquad \rho 2'(t) := \begin{pmatrix} \frac{d}{dt}\rho 2(t)_1 \\ \frac{d}{dt}\rho 2(t)_2 \\ \frac{d}{dt}\rho 2(t)_3 \end{pmatrix} \qquad \rho 3'(t) := \begin{pmatrix} \frac{d}{dt}\rho 3(t)_1 \\ \frac{d}{dt}\rho 3(t)_2 \\ \frac{d}{dt}\rho 3(t)_3 \end{pmatrix}$$

Then the perimeter is

$$L := \int_0^{3\cdot\sqrt{2}} \left|\rho 1'(t)\right|\,dt + \int_0^{3\cdot\sqrt{2}} \left|\rho 2'(t)\right|\,dt + \int_0^{\frac{\pi}{4}} \left|\rho 3'(t)\right|\,dt \qquad L = 43.701$$

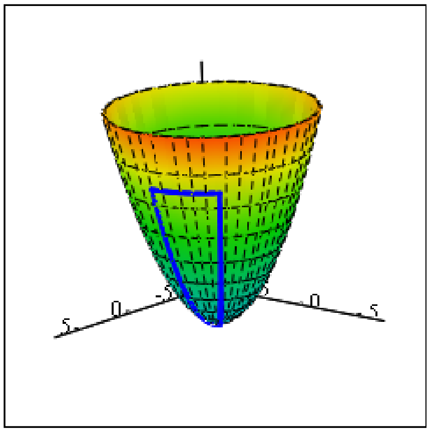
Finally, we define the tangent lines to the fist two curves at origin, which is their common point, and represent these lines. (See Figure 8.)

$$t1(s) := \rho1(0) + \rho1'(0)\cdot s \qquad t1 := \mathrm{CreateSpace}(t1, 0, 3)$$

$$t2(s) := \rho2(0) + \rho2'(0)\cdot s \qquad t2 := \mathrm{CreateSpace}(t2, 0, 4)$$

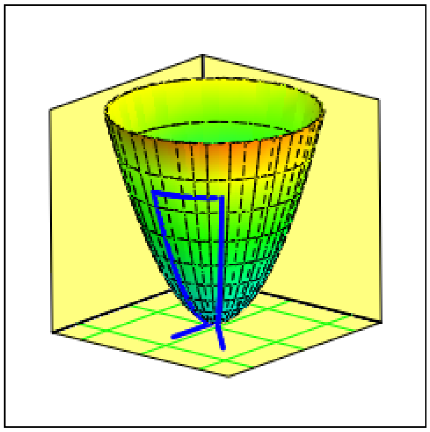
The angle between these curves at origin is

$$\theta := \mathrm{acos}\left(\frac{\rho1'(0)\cdot\rho2'(0)}{\left|\rho1'(0)\right|\cdot\left|\rho2'(0)\right|}\right) \qquad \theta \rightarrow \frac{1}{4}\cdot\pi$$



C1 , C2 , C3 , r

Figure 7



C1 , C2 , C3 , t1 , t2 , r

Figure 8

4. Conclusions

Differential Geometry is considered a difficult topic by the students in engineering because using it requires good skills in geometry, calculus and linear algebra. But the first difficulty for them is to "see" the surfaces and the curves in space.

The paper shows that using modern software like Mathcad the teacher can help the students to really see the surfaces and all the other elements related to them (coordinate curves, tangent planes, normal lines, arbitrary curves on surfaces etc.). By using Mathcad the teacher has a huge advantage: the equations are written in Mathcad similar to the blackboard. The students can easily see the connections between surfaces and their equations.

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